



TITLE:

Morse-Novikov numbers of 2-knots and surface-links (Intelligence of Low-dimensional Topology)

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CITATION:

Endo, Hisaaki ...[et al]. Morse-Novikov numbers of 2-knots and surface-links (Intelligence of Low-dimensional Topology). 数理解析研究所講究録 2016, 2004: 23-29: KJ00010275641.

ISSUE DATE:

2016-07

URL:

<http://hdl.handle.net/2433/231496>

RIGHT:

Morse-Novikov numbers of 2-knots and surface-links

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1. INTRODUCTION

1.1. A brief overview of the article. In this paper we give a short presentation of our results on the Morse-Novikov theory for 2-knots and surface-links (see the articles arXiv:1502.06352 and arXiv:1605.04532 for more details and full proofs.)

Let $N^k \subset S^{k+2}$ be a closed oriented submanifold, let $C(N) = S^{k+2} \setminus N$ be its complement. The orientation of N determines a cohomology class $\xi \in H^1(C(N)) \approx [C(N), S^1]$. We say that N is *fibred* if there is a Morse map $f : C(N) \rightarrow S^1$ homotopic to ξ which is regular nearby N (see Definition 1.1) and has no critical points. In general a Morse map $C(N) \rightarrow S^1$ has some critical points, the minimal number of these critical points will be called *the Morse-Novikov number of N* and denoted by $\mathcal{MN}(C(N))$.

In the first part of this paper we study this invariant in relation with constructions of spinning. The classical Artin's spinning construction [2] associates to each classical knot $K \subset S^3$ a 2-knot $S(K) \subset S^4$. A twisted version of this construction is due to E.C. Zeeman [12]. In [10] D. Roseman introduced a *frame spinning* construction, and G. Friedman [3] gave a generalization of D. Roseman's construction to include *twisting*. Let M be a framed closed submanifold of the $(m+k)$ -dimensional sphere, K be an m -knot and $\lambda : M \rightarrow S^1$ a C^∞ map. The twist spinning construction associates to these data an n -knot $\sigma(M, K, \lambda)$ (where $n = k + m$). In Section 2 we give an upper bound for the Morse-Novikov number of the twist spun knot in terms of Morse-Novikov invariants of M and K .

Section 3 is about Morse-Novikov theory for surface-links. In Subsection 3.1 we introduce a related invariant of surface-links, namely the *saddle number* $sd(F)$ (Definition 3.1) and prove the formula

$$(1) \quad \mathcal{MN}(C(F)) \leq 2sd(F) + \chi(F) - 2.$$

In Subsection 3.2 we discuss the case of spun knots. In subsection 3.3 we determine the Morse-Novikov numbers of certain surface-links.

1.2. Basic definitions. We start with the definition of a regular Morse map.

Definition 1.1. Let $N^k \subset S^{k+2}$ be a closed oriented submanifold. Denote by $\xi \in H^1(C(N)) \approx [C(N), S^1]$ the cohomology class dual to the orientation class of N . A Morse map $f : C(N) \rightarrow S^1$ is said to be *regular* if there is an orientation preserving C^∞ trivialisation

$$(2) \quad \Phi : T(N) \rightarrow N \times B^2(0, \epsilon)$$

of a tubular neighbourhood $T(N)$ of N such that the restriction $f|_{(T(N) \setminus N)}$ satisfies $f \circ \Phi^{-1}(x, z) = z/|z|$.

An f -gradient v of a regular Morse map $f : C(N) \rightarrow S^1$ will be called *regular* if there is a C^∞ trivialisation (2) such that $\Phi^*(v)$ equals $(0, v_0)$ where v_0 is the Riemannian gradient of the function $z \mapsto z/|z|$.

If f is a Morse map of a manifold to \mathbf{R} or to S^1 , then we denote by $m_p(f)$ the number of critical points of f of index p . The number of all critical points of f is denoted by $m(f)$.

Definition 1.2. The minimal number $m(f)$ where $f : C(N) \rightarrow S^1$ is a regular Morse map is called *the Morse-Novikov number of N* and denoted by $\mathcal{MN}(C(N))$.

To obtain lower bounds for numbers $m_p(f)$ one uses the *Novikov homology*. Let $L = \mathbb{Z}[t, t^{-1}]$; denote by $\hat{L} = \mathbb{Z}((t))$ and $\hat{L}_{\mathbb{Q}} = \mathbb{Q}((t))$ the rings of all series in one variable t with integer (respectively rational) coefficients and finite negative part. Recall that \hat{L} is a PID, and $\hat{L}_{\mathbb{Q}}$ is a field. Consider the infinite cyclic covering $\overline{C(N)} \rightarrow C(N)$; the Novikov homology of $C(N)$ is defined as follows:

$$\hat{H}_*(C(N)) = H_*(\overline{C(N)}) \otimes_{\hat{L}} \hat{L}.$$

The rank and torsion number of the \hat{L} -module $\hat{H}_k(C(N))$ will be denoted by $\hat{b}_k(C(N))$, respectively $\hat{q}_k(C(N))$. For any regular Morse function f there is a Novikov complex \mathcal{N}_* over \hat{L} generated in degree k by critical points of f of index k and such that $H_*(\mathcal{N}_*) \approx \hat{H}_*(C(N))$ (see [8]). Therefore we have the Novikov inequalities

$$\sum_k \left(\hat{b}_k(C(N)) + \hat{q}_k(C(N)) + \hat{q}_{k-1}(C(N)) \right) \leq \mathcal{MN}(C(N)).$$

These inequalities, which are far from being exact in general, are however very useful in the case of surface-links (see Section 3).

2. SPINNING AND RELATED CONSTRUCTIONS

2.1. Frame twist spun knots: the construction. In this subsection we recall the Artin-Zeeman-Roseman-Friedman frame twist spinning construction. The input data for this construction is:

- (TFS1) A closed manifold $M^k \subset S^{m+k}$ with trivial (and framed) normal bundle.
- (TFS2) An m -knot $K^m \subset S^{m+2}$.
- (TFS3) A C^∞ map $\lambda : M \rightarrow S^1$.

To these data one associates an n -knot $\sigma(M, K, \lambda)$, where $n = k + m$. When λ is a constant map we denote this knot by $\sigma(M, K)$; this is the Roseman's *frame spun knot*.

Let $a \in K^m$. Removing a small open disk $D(a)$ from S^{m+2} we obtain an embedded (knotted) disk K_0 in the disk $D^{m+2} \approx S^{m+2} \setminus D(a)$. We identify D^{m+2} with the standard Euclidean disk of radius 1 and center 0 in \mathbb{R}^{m+2} , then $\partial D^{m+2} = S^{m+1}$. We have the usual diffeomorphism

$$\chi : S^{m+1} \times]0, 1] \xrightarrow{\approx} D^{m+2} \setminus \{0\}, \quad (x, t) \mapsto tx.$$

We can assume that $K_0 \cap \partial D^{m+2}$ is an equatorial sphere $^\dagger S^{m-1}$ in $\partial D^{m+2} = S^{m+1}$. Moreover, we can assume that the intersection of K_0 with a neighbourhood of ∂D^{m+2} is also standard, that is,

$$K_0 \cap \chi(S^{m+1} \times [1 - \epsilon, 1]) = \chi(S^{m-1} \times [1 - \epsilon, 1]).$$

We have a framing of M in S^n (recall that $n = m + k$); combining this with the standard framing of S^n in S^{n+2} we obtain a diffeomorphism

$$\Phi : N(M, S^{n+2}) \xrightarrow{\approx} M \times D^m \times D^2$$

where $N(M, S^{n+2})$ is a regular neighbourhood of M in S^{n+2} . We can assume that the restriction of Φ to $N(M, S^n)$ is a diffeomorphism

$$\Phi : N(M, S^n) \xrightarrow{\approx} M \times D^m \times \{0\}$$

induced by the given framing of M . The Euclidean disc D^{m+2} is a subset of $D^m \times D^2$, so that $K_0 \subset D^m \times D^2$.

For $\theta \in S^1$ denote by R_θ the rotation of D^2 around its center. The disc $D^{m+2} \subset D^m \times D^2$ is invariant with respect to this rotation as well as the intersection of K_0 with a small neighbourhood of ∂D^{m+2} . We have $\Phi(S^n \cap N(M, S^{n+2})) = M \times D^m \times \{0\}$. Let

$$Z = \{(x, y, z) \mid (y, z) \in R_{\lambda(x)}(K_0)\}.$$

This is an n -dimensional submanifold of $M \times D^m \times D^2$. We define $\sigma(M, K, \lambda)$ as follows

$$\sigma(M, K, \lambda) = \left(S^{n+2} \setminus N(M, S^{n+2}) \right) \cup \Phi^{-1}(Z).$$

This is the image of an embedded n -sphere, knotted in general.

Examples and particular cases.

- 1) Let $\dim M = 0$, so that M is a finite set; denote by p its cardinality. Then the n -knot $\sigma(M, K, \lambda)$ is equivalent to the connected sum of p copies of K .
- 2) If M is the equatorial circle of the sphere S^2 , which is in turn considered as an equatorial sphere of S^4 , and $\lambda(x) = 1$ for all x , we obtain the classical Artin's construction. If $\lambda : S^1 \rightarrow S^1$ is a map of degree d , we obtain the Zeeman's twist-spinning construction [12].
- 3) If $\lambda(x) = 1$ for all $x \in M$ we obtain the Roseman's construction of spinning around the manifold M [10]. In this case we will denote $\sigma(M, K, \lambda)$ by $\sigma(M, K)$.

2.2. Morse-Novikov numbers of twist spun knots.

Theorem 2.1.

$$\mathcal{MN}\left(C(\sigma(M, K, \lambda))\right) \leq \mathcal{MN}(C(K)) \cdot \mathcal{MN}(M, [\lambda]).$$

(where $[\lambda] \in H^1(M, \mathbb{Z}) \approx [M, S^1]$ is the homotopy class of λ).

[†]By *equatorial sphere* in $S^N \subset \mathbb{R}^{N+1}$ we mean the intersection of a linear subspace $L \subset \mathbb{R}^{N+1}$ with S^N ; this intersection is a Euclidean sphere of dimension $\dim L - 1$.

Corollary 2.2. *Let $K \subset S^3$ be a classical knot, denote by $S(K)$ the spun knot of K . Then*

$$(3) \quad \mathcal{MN}(C(S(K))) \leq 2\mathcal{MN}(C(K)).$$

Proof. In this case $M = S^1$ and $[\lambda] = 0$. We have $\mathcal{MN}(S^1, 0) = 2$ and the result follows. \square

The classical theorems concerning fibrations of spun knots follow from Theorem 2.1:

Corollary 2.3. *(D. Roseman [10]) If K is fibred, then $\sigma(M, K)$ is fibred.*

Proof. Since $\mathcal{MN}(C(K)) = 0$, Theorem 2.1 implies $\mathcal{MN}(C(\sigma(M, K))) = 0$. \square

Corollary 2.4. *(E.C. Zeeman [12]) The d -twist spun knot of any classical knot K is fibred for $d \geq 1$.*

Proof. Let Σ be an equatorial circle in S^2 . The d -twist spun knot of K is by definition the 2-knot $\sigma(\Sigma, K, \lambda)$ in S^4 where $\lambda : \Sigma \rightarrow \Sigma$ is a map of degree d . The assertion follows, since $\mathcal{MN}(S^1, [\lambda]) = 0$. \square

Remark 2.5. The Zeeman's theorem above generalizes immediately to the following statement: If $\mathcal{MN}(M, [\lambda]) = 0$, then the knot $\sigma(M, K, \lambda)$ is fibred for any knot K .

2.3. Rotation. In this subsection we present one more geometric construction related to spinning techniques. Let Σ be an equatorial n -sphere of S^{n+1} . We can view the sphere S^{n+1} as the union of two $(n+1)$ -dimensional discs $D_+ \cup D_-$ intersecting by Σ . Consider S^{n+1} as the equatorial sphere of S^{n+2} . The sphere S^{n+2} can be considered as the result of rotation of the disc D_+ around its boundary Σ . We have the (linear orthogonal) action of S^1 on S^{n+2} , such that Σ is the fixed point set of the action, and the action is free on the rest of the sphere S^{n+2} . Let K^{n-1} be an $(n-1)$ -knot in S^{n+1} . We can assume that $K^{n-1} \subset \text{Int } D_+$. Rotation of K^{n-1} around Σ gives a submanifold $R(K)$ of codimension 2 in S^{n+2} . The manifold $R(K)$ is diffeomorphic to $S^1 \times K$. We call this construction *rotation*. When $\dim K = 1$, the manifold $R(K)$ is sometimes called the *spun torus* of K . In this section we relate the Morse-Novikov numbers of $R(K)$ with those of K .

Theorem 2.6.

$$\mathcal{MN}(C(R(K))) \leq 2\mathcal{MN}(C(K)) + 2.$$

3. MORSE-NOVIKOV NUMBERS OF SURFACE-LINKS

In this section we develop circle-valued Morse theory for surface-links.

3.1. Motion pictures and saddle numbers. Let F be a surface-link, that is, a closed oriented 2-dimensional C^∞ submanifold of S^4 . We can assume $F \subset \mathbb{R}^4$.

Choose a projection p of \mathbb{R}^4 onto a line. Assume that the critical points of the function $p|_F$ are non-degenerate. Denote by $sdl(F)$ the minimal number of saddle points of $p|_F$ over all the projections p .

Definition 3.1. A saddle number $sd(F)$ is the minimum of numbers $sdl(F')$ where F' ranges over all surface-links F' ambiently isotopic to F .

The invariant $sd(F)$ is closely related to the *ch-index* of F , introduced and studied by K. Yoshikawa in [11]. In particular, we have $sd(F) \leq ch(F)$. In order to relate the number $sd(F)$ to $\mathcal{MN}(S^4 \setminus F)$ we will reformulate the definition of the saddle number.

Let $F \subset S^4$ be a surface-link. The equatorial 3-sphere Σ^3 of the standard Euclidean sphere S^4 divides S^4 into two parts:

$$S^4 = D_+^4 \cup D_-^4, \quad \text{with } D_+^4 \cap D_-^4 = \Sigma^3.$$

We assume that F is included in $\text{Int}(D_-^4)$ and F does not contain the centre of D_-^4 . Perturbing the embedding $F \subset D_-^4$ if necessary, we can assume that the restriction $\rho = r|_F$ of the radius function $r : D_-^4 \rightarrow [0, 1]$ is a Morse function. The family $\{(r^{-1}(t), \rho^{-1}(t))\}_{t \in [0, 1]}$ of possibly singular links can be drawn as a *motion picture* (see [5], Chapter 8). Each singularity of a link in the family corresponds to a critical point of ρ . A critical point of ρ of index 0 (1, 2, respectively) is called *minimal point* (*saddle point*, *maximal point*, respectively) of ρ , which is represented by a *minimal band* (*saddle band*, *maximal band*, respectively) in (a modification of) the motion picture.

It is clear that the minimal number of the saddle points for all such Morse functions ρ and all surface-links ambiently isotopic to F is equal to $sd(F)$.

Theorem 3.2. $\mathcal{MN}(C(F)) \leq 2sd(F) + \chi(F) - 2$.

Corollary 3.3. Let $K \subset S^4$ be a 2-knot. Then $\mathcal{MN}(C(K)) \leq 2sd(K)$. \square

Proposition 3.4. Let $F \subset S^4$ be the trivial k -component surface-link. Then $\mathcal{MN}(C(F)) = 4k - 2 - \chi(F)$.

Proof. It is not difficult to show that $\widehat{b}_1(C(F)) \geq k - 1$, $\widehat{b}_3(C(F)) \geq k - 1$. Therefore for every regular Morse map $f : C(F) \rightarrow S^1$ we have $m_1(f) + m_3(f) \geq 2(k - 1)$. Assuming $m_0(f) = m_4(f) = 0$ we have $m_1(f) - m_2(f) + m_3(f) = 2 - \chi(F)$, and $\mathcal{MN}(C(F)) \geq 4k - 2 - \chi(F)$; this lower bound coincides with the upper bound derived from Theorem 3.2. \square

3.2. Spun knots. Let K be a classical knot in S^3 ; denote by $S(K)$ the corresponding spun knot.

Proposition 3.5. If K is a non-fibered knot of tunnel number 1, then $\mathcal{MN}(S^4 \setminus S(K)) = 4$.

Proof. Recall that $\mathcal{MN}(S^4 \setminus S(K)) \leq 2\mathcal{MN}(K)$ (Corollary 2.2). In the paper [7] of the second author it is shown that $\mathcal{MN}(C(K)) \leq 2t(K)$, hence $\mathcal{MN}(C(S(K))) \leq 4$ by Corollary 2.2. Put $G = \pi_1(S^3 \setminus K)$, then $\pi_1(S^4 \setminus S(K)) \approx G$; let $H = [G, G]$. Let $f : S^4 \setminus S(K) \rightarrow S^1$ be a regular Morse map without minima and maxima. If $m_1(f) = 0$, then a standard Morse-theoretic argument applied to the infinite cyclic cover of $S^4 \setminus S(K)$ implies that H is finitely generated, which is impossible, since K is not fibred. Therefore $m_1(f) \geq 1$, and similarly, $m_3(f) \geq 1$, hence $m_2(f) \geq 2$ and the proposition is proved. \square

3.3. Surface-links of Yoshikawa's table. A. Kawauchi, T. Shibuya and S. Suzuki [6] developed a method of representing surface-links by diagrams. Based on this method K. Yoshikawa [11] introduced a numerical invariant $ch(F)$ of surface-links F and enumerated all the (weakly prime) surface-links F with $ch(F) \leq 10$.

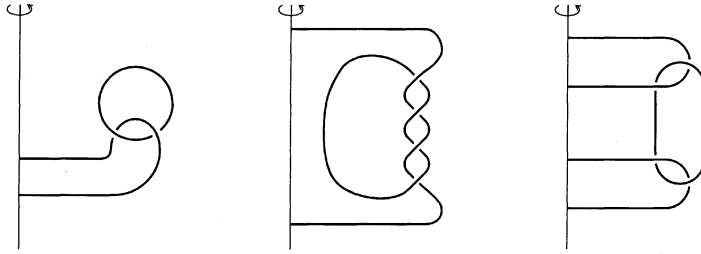


FIGURE 1

It is clear from the definition of the invariant $ch(F)$ that we have $sd(F) \leq ch(F)$. In the rest of this section we assume that the reader is familiar with Yoshikawa's work, and with his terminology. There are 6 two-knots in Yoshikawa's table, namely

$$0_1, 8_1, 9_1, 10_1, 10_2, 10_3.$$

The trivial 2-knot 0_1 is obviously fibred. The knots 8_1 and 10_1 are spun knots of the trefoil knot and respectively of the figure 8 knot, thus both 8_1 and 10_1 are fibred by [1].

The case of 9_1 is more complicated. The saddle number of this 2-knot is 2. Therefore $\mathcal{MN}(9_1) \leq 4$. Using the presentation of the fundamental group of the complement to 9_1 (see [11]) and Poincaré duality properties it is easy to compute the Novikov numbers of 9_1 . Namely we have $\hat{q}_1 = 1, \hat{q}_2 = \hat{q}_3 = 0$. Therefore

$$2 \leq \mathcal{MN}(9_1) \leq 4.$$

The 2-knot 10_2 is the 2-twist-spun knot of the trefoil knot, hence fibred by Zeeman's theorem [12]. Similarly, 10_3 is fibred, being the 3-twist spun of the trefoil knot.

The surface-link $6_1^{0,1}$ is the result of spinning of the Hopf link which is fibred (see the left of Figure 2) therefore $\mathcal{MN}(6_1^{0,1}) = 0$.

The surface-link $8_1^{1,1}$ is the spun torus of the Hopf link. Applying Theorem 2.6 we get the upper bound $\mathcal{MN}(8_1^{1,1}) \leq 2$. Computing the Euler characteristic implies the inverse inequality, so $\mathcal{MN}(8_1^{1,1}) = 2$.

The same argument applies to the surface-link 10_1^1 , which is the spun torus of the trefoil knot, see the figure 2 (middle), so that $\mathcal{MN}(10_1^1) = 2$.

The surface-link $10_1^{0,1}$ is the result of spinning of the link 4_1^2 which is fibred, therefore $\mathcal{MN}(10_1^{0,1}) = 0$.

The case of the surface-link $F = 10_1^{0,0,1}$ is more complicated. Applying a generalisation of spinning constructions we prove that $\mathcal{MN}(10_1^{0,0,1}) = 2$.

4. ACKNOWLEDGEMENTS

This work was accomplished when the second author was visiting the Tokyo Institute of Technology in 2016 with the support of the JSPS fellowship. The first author was partially supported by JSPS KAKENHI Grant Numbers 25400082, 16K05142. The second author thanks the Tokyo Institute of Technology for support and warm hospitality.

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